Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 261–267, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00033-0

INDENTATION WITH ADHESION OF A SYMMETRICAL PUNCH INTO AN ELASTIC HALF-PLANE†

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(*Received* 15 *March* 1995)

A new method of solving the problem of the adhesive penetration of a punch into an elastic half-plane is proposed, based on the well-known [1] procedure of inverting the equations for the contact stresses. The problem is considered in the refined formulation, which takes into account the tangential displacement of the boundary in the boundary condition for the normal displacement. A similar problem was solved previously in [2–4] using the incremental approach, which requires additional transformations of the initial equations. © 1996 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Suppose a rigid symmetrical punch is indented into an elastic half-plane by a central applied load P when there is adhesion between the contacting bodies (see Fig. 1). The latter denotes that points of the half-plane which get into contact with the punch are displaced further together with it vertically.

We will assume that during indentation, the dimension $a > 0$ of the contact region increases monotonically, and we will characterize the degree of penetration of the punch by the quantity a. The problem is to find the contact stresses $q_1 = \tau_{xy}|_{y=0}, q_2 = -\sigma_y|_{y=0}$ for arbitrary a. The boundary conditions have the form

$$u(x,a) = \varphi(x),\ v(x,a) = g(x+\mu\varphi(x)),\ |x| \leqslant a \tag{1.1}$$
$$q_1(x,a) = q_2(x,a) = 0,\ |x| > a$$

where u and v are the tangential and normal displacements of the boundary of the half-plane in the x, y system (see Fig. 1), $\varphi(x)$ is a certain function to be determined, $y = g(x)$ is the equation of the shape of the punch, where, in view of the symmetry of the problem, the function $g(x)$ is even, and $\mu = 0$ or $\eta = 1$. Henceforth we will also use the notation $G(x) = g'(x)$.

We emphasize that the function $\varphi(x)$ is the tangential boundary displacement within the region of contact, and the fact that the argument a does not occur in it is due to the fact that this displacement is independent of the parameter a when the punch adheres to the half-plane.

The presence of the term $\varphi(x)$ in the argument of the function g in (1.1) when $\mu = 1$ corresponds to the more refined formulation of the contact problem, which takes the tangential displacement of the boundary into account in the boundary condition for v.‡ When the term $\varphi(x)$ does not occur in the second equation of (1.1) ($\mu = 0$) the latter will have the classical form [3].

We will note some properties of the function $\varphi(x)$. Firstly, in view of the symmetry of the problem, this function must be odd and

$$\varphi(0) = 0 \tag{1.2}$$

Further, since $\varphi(x)$ is the tangential displacement of the point x of the region of contact, the inequality $x + \varphi(x) < 0$ when $x \geqslant 0$ denotes that when the half-plane is deformed the point x is displaced through the centre of the punch to the left, and this obviously contradicts the condition for the contacting surfaces of the half-plane and the punch to adhere. Hence it follows that

$$x + \varphi(x) \geqslant 0 \quad \text{for} \quad x \geqslant 0 \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 267–273, 1996.

‡SOLDATENKOV I. A., The contact problem for a half-plane in a more refined formulation (taking the tangential contact displacement into account). Preprint No. 501. Institute for Problems in Mechanics, Academy of Sciences of the USSR, Moscow, 1991.

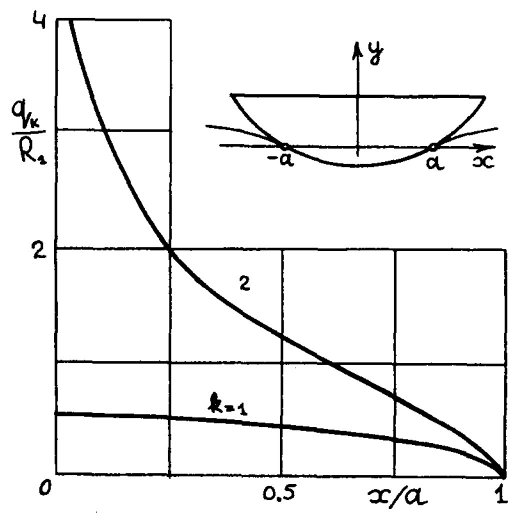

Fig. 1.

In our further discussion we will use the idea of a node and classes of functions H, H_0, H^* and h_2, defined in [1]. We will introduce the class of functions H_+, which we define as $f(x) \in H_+$, if, when $f(0) = \lim_{x \to 0+0} f(x)$ and for any $d > 0$, the function $f(x)$ belongs to H in the section $[0, d]$.

We will impose the following limitations on the shape of the punch and the unknown function $\varphi(x)$

$$G(x) \equiv g'(x) \in H_+, \quad \varphi'(x) \in H_+ \tag{1.4}$$

Note that if $f(x) \in H_+$ and the function $f(x)$ is even or odd (like the functions $\varphi'(x)$ and $g'(x)$) and if we consider the point $x = 0$ as the node, we obtain $f(x) \in H_0$ in $[-d, d]$ for any $d > 0$. Taking this into account, conditions (1.4) can be written in the form

$$G(x) \in H_0, \quad \varphi'(x) \in H_0 \quad \text{in } [-d, d], \ d > 0 \tag{1.5}$$

where the point $x = 0$ is regarded as a node.

We will now consider the boundary stresses q_1 and q_2, the connection of which with the displacements u and v is given, in the linear theory of elasticity, by two singular integral equations [5]. Taking boundary conditions (1.1) into account these equations can be represented in the form

$$\begin{aligned} \psi_1(x) &= -\pi\chi q_2(x,a) + \int_{-a}^{a} \frac{q_1(\xi,a)}{\xi - x} d\xi, \quad |x| \leqslant a \\ \psi_2(x) &= -\pi\chi q_1(x,a) - \int_{-a}^{a} \frac{q_2(\xi,a)}{\xi - x} d\xi, \quad |x| \leqslant a \end{aligned} \tag{1.6}$$

where

$$\psi_1(x) = m\varphi'(x), \quad \psi_2(x) = mG(x + \mu\varphi(x))(1 + \mu\varphi'(x)) \tag{1.7}$$

$$m = \frac{\pi E}{2(1 - \nu^2)}, \quad \chi = \frac{1 - 2\nu}{2(1 - \nu)}$$

E is Young's modulus and ν is Poisson's ratio, $0 \leqslant \nu \leqslant 1/2$.

It can be shown that when conditions (1.5) are satisfied the following property of the functions ψ_1 and ψ_2 follows from (1.7)

$$\psi_k(x) \in H_0 \quad \text{in } [-d, d], \ d > 0, \ k = 1, 2 \tag{1.8}$$

As regards the functions $q_1(x, a)$ and $q_2(x, a)$ we will assume that for each a they are bounded at the ends $\pm a$ of the contact region and belong to the class H^* in $[-a, a]$, i.e. in the notation employed in [1] we will assume that

$$q_k(x,a)\in h_2 \ \text{ in } [-a,a],\ k=1,2 \tag{1.9}$$

2. INVERSION OF THE EQUATIONS FOR THE BOUNDARY STRESSES

We fix the dimension a of the contact region. We can invert system (1.6), i.e. obtain expressions for q_1 and q_2 in terms of u and v, using the well-known results for systems of singular integral equations [6]. However, in this case it is simpler to reduce system (1.6) to a single complex-valued singular integral equation, by the method described, for example, in [7], and use the method from [1] to solve it. Thus, we multiply the first equation in (1.6) by i and add it to the second equation. We obtain

$$\begin{gathered}-\pi\chi q(x,a)+i\int_{-a}^{a}\frac{q(\xi,a)}{\xi-x}d\xi=f(x),\ |x|\leqslant a\\ q(x,a)=q_1(x,a)+iq_2(x,a),\ f(x)=\psi_2(x)+i\psi_1(x)\end{gathered}\tag{2.1}$$

It follows directly from (1.8) and (1.9) that

$$f(x)\in H_0,\ q(x,a)\in h_2 \ \text{ in } [-a,a]$$

By [1], a solution $q(x, a)$ of Eq. (2.1) of class h_2 when $f(x)\in H_0$ exists if and only if

$$\int_{-a}^{a}\frac{f(\xi)d\xi}{Z_1(\xi,a)}=0 \tag{2.2}$$

where

$$Z_1(x,a)=(a^2-x^2)^{1/2}\left(\frac{a-x}{a+x}\right)^{i\tau/2},\quad \tau=\frac{1}{\pi}\ln\frac{1+\chi}{1-\chi}\geqslant 0$$

When condition (2.2) is satisfied the following formula for the inversion of Eq. (2.1) holds

$$\begin{gathered}q(x,a)=A^*f(x)-\frac{B^*}{\pi i}Z_1(x,a)\int_{-a}^{a}\frac{f(\xi)d\xi}{Z_1(\xi,a)(\xi-x)},\ |x|\leqslant a\\ A^*=\frac{\chi}{\pi(1-\chi^2)},\quad B^*=\frac{1}{\pi(1-\chi^2)}\end{gathered}\tag{2.3}$$

Note that the limit values of the function $q(x, a)$ from (2.3) as $x\to\pm a$ are equal to zero. This follows directly from the well-known theorem of the behaviour of the Cauchy-type integral in (2.3) in the region of the ends of the integration line [1].

We substitute into (2.2), instead of the function $f(x)$, its expression in terms of the functions $\psi_1(x)$ and $\psi_2(x)$, which, in turn, are related to the functions $\varphi(x)$ and $g(x)$ by Eqs (1.7). As a result, taking into account the symmetry properties of the functions $\varphi(x)$ and $g(x)$, we obtain the equation

$$\begin{gathered}\int_0^a\frac{\varphi'(x)\cos\alpha(x,a)+G(x+\mu\varphi(x))(1+\mu\varphi'(x))\sin\alpha(x,a)}{(a^2-x^2)^{1/2}}dx=0\\ \alpha(x,a)=\frac{\tau}{2}\ln\frac{a+x}{a-x}\end{gathered}\tag{2.4}$$

Equation (2.4) must be satisfied for any $a>0$ and hence, can be regarded as a Volterra equation of the first kind for the unknown function $\varphi(x)$.

Making a similar substitution into (2.3) we will have for $x \in [-a, a]$

$$q_1(x,a) = \frac{1}{\pi(1-\chi^2)}\Big\{\chi\psi_2(x) - \frac{1}{\pi}(a^2 - x^2)^{1/2}[i_1(x,a)\sin\alpha(x,a) +$$
$$+ j_1(x,a)\cos\alpha(x,a) + i_2(x,a)\cos\alpha(x,a) - j_2(x,a)\sin\alpha(x,a)]\Big\}$$
$$q_2(x,a) = \frac{1}{\pi(1-\chi^2)}\Big\{\chi\psi_1(x) - \frac{1}{\pi}(a^2 - x^2)^{1/2}[i_1(x,a)\cos\alpha(x,a) -$$
$$- j_1(x,a)\sin\alpha(x,a) - i_2(x,a)\sin\alpha(x,a) - j_2(x,a)\cos\alpha(x,a)]\Big\} \tag{2.5}$$
$$i_k(x,a) = \int_{-a}^{a} \frac{\psi_k(\xi)\sin\alpha(\xi,a)}{(a^2-\xi^2)^{1/2}(\xi - x)}d\xi, \quad j_k(x,a) = \int_{-a}^{a} \frac{\psi_k(\xi)\cos\alpha(\xi,a)}{(a^2-\xi^2)^{1/2}(\xi - x)}d\xi$$

Expressions (2.5) for $q_k(x, a)$ can be given a more compact form if we make the change of variables $x \to s = \operatorname{arcth}(x/a)$ in them and introduce the functions

$$Q_k(s,a) = q_k(a \operatorname{th} s, a), \quad \Psi_k(s,a) = \psi_k(a \operatorname{th} s) \tag{2.6}$$

Equations (2.5) give the solution of the problem in question, when the function $\varphi(x)$ is present, which satisfies Eq. (2.4).

For known contact stresses q_1 and q_2, the load P on the punch is found from its condition of equilibrium

$$P = -2\int_0^a q_1(x,a)\sin\omega(x)dx + 2\int_0^a q_2(x,a)\cos\omega(x)dx \tag{2.7}$$

where $\omega(x) = \operatorname{arctg} g'(x)$ is the angle of inclination of the contour of the punch to the x axis, $x \geqslant 0$, where $|\omega(x)| \ll 1$.

Note. When $\nu = 1/2$ (an incompressible material) we have $\tau = 0$, and, consequently, Eq. (2.4) in $\varphi(x)$ will have the form

$$\int_0^a \frac{\varphi'(x)dx}{(a^2 - x^2)^{1/2}} = 0, \quad a > 0 \tag{2.8}$$

The function $\varphi(x) \equiv 0$ obviously satisfies Eq. (2.8) when condition (1.2) is satisfied. It can be proved that the solution $\varphi(x) \equiv 0$ of Eq. (2.8) is unique. To do this it is sufficient to reduce it to the equation†

$$\varphi(a) = a^{-1}\int_0^a N_0\left(\frac{x}{a}\right)\varphi(x)dx, \quad a > 0$$
$$N_0(\lambda) = \frac{2\sqrt{2}}{\pi}\frac{\lambda}{(1+\lambda)^{3/2}}D\left(\sqrt{\frac{1-\lambda}{1+\lambda}}\right) \tag{2.9}$$

where $D(k)$ is the complete elliptic integral, and it can be shown that the integral operator on the right-hand side of (2.9) is contractive in the space of the continuous functions $C[0, d]$ for any $d > 0$.†

Substituting the function $\varphi(x) \equiv 0$ into (2.5) we obtain expressions for q_1 and q_2 for $\nu = 1/2$, which have the same form as in the case of the contact problem without friction

$$q_1(x,a) \equiv 0, \quad q_2(x,a) = \frac{2E}{3\pi}(a^2 - x^2)^{1/2}\int_{-a}^{a}\frac{g'(\xi)d\xi}{(a^2-\xi^2)^{1/2}(\xi - x)}$$

Below we consider a special case of a wedge-shaped punch, for which, by Eqs (2.4) and (2.5), the solution of the problem can be found in explicit form. Some other special cases were considered in the reference mentioned in the footnote.

†SOLDATENKOV I. A., The problem of the indentation with adhesion of a punch into an elastic half-space. Preprint No. 525. Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 1993.

3. THE CASE OF A WEDGE-SHAPED PUNCH

Suppose

$$g(x)=g_1|x|, \quad G(x)=g_1 \operatorname{sgn} x, \quad g_1>0 \tag{3.1}$$

Note that, to satisfy the condition for the deformations to be small, which is necessary for the use of Eqs (1.6) of the linear theory of elasticity to be justified, we must assume $g_1 \ll 1$.

It can be established that when (3.1) holds, the first of conditions (1.4) is satisfied. Substituting (3.1) into (1.7) and (2.4) and taking inequality (1.3) into account, we arrive at the following expressions for ψ_1 and ψ_2 and the equation for $\varphi(x)$

$$\psi_1(x)=m\varphi'(x), \quad \psi_2(x)=mg_1 \operatorname{sgn} x(1+\mu\varphi'(x)) \tag{3.2}$$

$$\int_0^a \frac{\cos\alpha(x,a)+\mu g_1 \sin\alpha(x,a)}{(a^2-x^2)^{1/2}}\varphi'(x)dx=-g_1\int_0^a \frac{\sin\alpha(x,a)}{(a^2-x^2)^{1/2}}dx \tag{3.3}$$

The following function satisfies (3.3)

$$\varphi'(x)=\varphi_1 \equiv -g_1\gamma_0/(\delta_0+\mu g_1\gamma_0), \quad x \geqslant 0 \tag{3.4}$$

where

$$\gamma_0 \equiv \int_0^\infty \frac{\sin\tau X}{\operatorname{ch} X}dX=2\tau\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)^2+\tau^2} \geqslant 0$$

$$\delta_0 \equiv \int_0^\infty \frac{\cos\tau X}{\operatorname{ch} X}dX=\frac{\pi}{2}\left(\operatorname{ch}\frac{\pi\tau}{2}\right)^{-1} \geqslant 0$$

Bearing in mind Eq. (1.2) and also the fact that the function $\varphi(x)$ is odd, we obtain from (3.4)

$$\varphi(x)=\varphi_1 x \tag{3.5}$$

Expression (3.5) for $\varphi(x)$ obviously satisfied the second of conditions (1.4). Moreover, inequality (1.3) is satisfied for a function of the form (3.5). When $\mu = 1$ this inequality follows from (3.4) and (3.5) and when $\mu = 0$ it also follows from the inequality

$$1-g_1\gamma_0/\delta_0>0 \tag{3.6}$$

the correctness of which is due to the fact that $g_1 \ll 1$ and, as can be shown, $\gamma_0/\delta_0 < 1$ when $0 \leqslant \nu \leqslant 1/2$.

If we substitute (3.2) and (3.5) into (2.5), then, after simple reduction of the integrals i_k and j_k, we can obtain the following expressions for the functions $Q_1(x, a)$ and $Q_2(x, a)$, related to the constant stresses $q_1(x, a)$ and $q_2(x, a)$ by Eqs (2.6)

$$Q_k(s,a)=R_1 W_k(s), \quad s>0, \quad R_1=\frac{4(1-\nu)Eg_1(1+\mu g_1\gamma_0/\delta_0)^{-1}}{\pi(1+\nu)(3-4\nu)} \tag{3.7}$$

$$W_1(s)=\int_s^\infty \frac{\sin\tau X}{\operatorname{sh} X}dX, \quad W_2(s)=\int_s^\infty \frac{\cos\tau X}{\operatorname{sh} X}dX \tag{3.8}$$

We will analyse the behaviour of the contact stresses as $x \to a$ ($s \to \infty$) and $x \to 0+0$ ($s \to 0+0$). To do this we expand the function $(\operatorname{sh} X)^{-1}$ in a power series and integrate the sums obtained term by term. We obtain the following equations

$$W_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(s)=2\sum_{n=1}^\infty \frac{e^{-(2n-1)s}}{(2n-1)^2+\tau^2}\left[(2n-1)\begin{Bmatrix}\sin\tau s\\ \cos\tau s\end{Bmatrix} \pm \tau\begin{Bmatrix}\cos\tau s\\ \sin\tau s\end{Bmatrix}\right], \quad s>0 \tag{3.9}$$

by means of which, and also relations (2.6), it can be established that as $x \to a$ ($s \to \infty$)

$$q_k(x,a) = \frac{2R_1}{(1+\tau^2)^{1/2}}\left(\frac{a-x}{a+x}\right)^{1/2}\sin\left(\frac{\tau}{2}\ln\frac{a+x}{a-x}+\theta_k\right)+O((a-x)^{3/2}),\quad x\to a,\quad k=1,2 \tag{3.10}$$

$$\theta_k = \arccos(1+\tau^2)^{-1/2}+\pi(k-1)/2$$

Note that, as also for a parabolic punch [3], the boundary stresses q_k tend to zero as $x \to a$ by a square-root law with an infinite number of oscillations.

To analyse the behaviour of q_k as $x \to 0+0$ we must consider the behaviour of the integrals (3.8) as $s \to 0+0$. Analysis shows that

$$W_1(s) = \frac{\pi}{2}\chi + O(s),\quad W_2(s) = -\ln s + O(1),\quad \chi = \operatorname{th}\frac{\pi\tau}{2}$$

when $s \to 0+0$, and hence, taking (2.6) and (3.7) into account we obtain

$$q_1(x,a) = \frac{\pi}{2}R_1\chi + O\left(\frac{x}{a}\right),\quad q_2(x,a) = -R_1\ln\frac{x}{a}+O(1) \tag{3.11}$$

Relations (3.10) and (3.11) define the nature of the behaviour of the stresses q_k in the neighbourhood of the points $x = a$ and $x = 0$. Intermediate values of q_k (i.e. for $x \in (0, a)$) can be calculated using the chain of equalities (3.9), (3.7) and (2.6). The corresponding result for $D = 0.25$ is shown in Fig. 1.

From (2.6) (3.7) and (3.8), which define the contact stresses q_k, the condition of equilibrium (2.7) gives the following expression for the load on a wedge-shaped punch

$$P = 2R_1\delta_0\cos\omega\left(1-g_1\frac{\gamma_0}{\delta_0}\right)a \tag{3.12}$$

Note that, in view of (3.6), the expression in parentheses on the right-side of (3.12) and, consequently, the expression for P itself, take only positive values.

4. CONCLUDING REMARKS

When using the classical boundary conditions (1.1) (i.e. for $\mu = 0$) Eq. (2.4) becomes linear and can be solved in explicit form for the more general case of a symmetrical punch

$$g(x) = \sum_{n=0}^{N} g_n x^n,\quad x \geqslant 0$$

It can be verified, by direct substitution into (2.4) with $\mu = 0$, that the corresponding solution has the form

$$\varphi(x) = -\sum_{n=1}^{N}\frac{\gamma_{n-1}}{\delta_{n-1}}g_n x^n,\quad x \geqslant 0$$

where γ_n, δ_n are constant coefficients.

We recall that $\varphi(x)$ is the tangential displacement of points in the contact region. Using the results obtained in [3] one can also obtain an expression for $\varphi(x)$ in the case of a polynomial punch, which will be of the same accuracy as that presented above.

It can be verified, for a wedge-shaped punch, that the solution of the corresponding problem obtained for classical boundary conditions [3], can be generalized to the case of boundary conditions of the form (1.1) with $\mu = 0.1$. The expressions obtained for Q_k as a result have the same forms as (3.7). Hence, for a wedge-shaped punch the methods described above and in [3] give the same results. However, in general, it is difficult to establish that both methods are equivalent.

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Translated by R.C.G.